Prof. Amador Martin-Pizarro — Albert-Ludwigs-Universität Freiburg
Übungen: Xier Ren — Sommersemester 2024

Topology

Problem Sheet 11

Deadline: 9 July 2024, 15h (**This is the last sheet in this semester**).

Exercise 1 (4 Points)**.**

Show that the map
$$r: \begin{array}{ccc} \mathbb{R}^{n+1}\setminus\{\bar{0}\} & \to & \mathbb{S}^n \\ \bar{x} & \mapsto & \frac{\bar{x}}{|\bar{x}|} \end{array}$$
is a deformation retraction.

Are $\mathbb{R}^2$ and $\mathbb{R}^3$ homeomorphic?

Exercise 2 (4 Points)**.**

Show that every continuous map $f:\mathbb{D}\to\mathbb{D}$ such that $f(x)=x$ for $x\in\mathbb{S}^1$ must be surjective.
Hint: Brouwer's fixed point theorem.

Exercise 3 (6 Points)**.**

Consider a finite group H with the discrete topology and a topological space X. An action of H on X is *continuous* if the action map
$$\begin{array}{ccc} H\times X & \to & H \\ (h,x) & \mapsto & h\star x \end{array}$$
is continuous with respect to the product topology on $H\times X$.

a) Suppose that X is Hausdorff and $\mathrm{Fix}_X(h)=\emptyset$ for all $h\neq 1_H$ in H. Show that for x in X, there is a open neighbourhood U of x with $h_1\star U$ and $h_2\star U$ disjoint for all $h_1\neq h_2$ in H.

b) If X and H are as in a), show that the quotient map $X\to X_H$ is a covering, where X_H is the collection of all H-orbits of X under the action of H equipped with the quotient topology.

c) Deduce that for every topological group G such that $\{1_G\}$ is a closed subgroup and every finite subgroup H of G, the quotient map $G\to G/H$ is a covering, where G/H consists of all H-classes of elements of G.

Exercise 4 (6 Points)**.**

a) Given a non-empty subset A of the metric space (X,d), show that the distance function $\mathrm{dist}(\cdot,A): x\mapsto \inf_{y\in A}\mathrm{d}(x,y)$ is well-defined and continuous.

   If $\mathrm{dist}(x,A)=0$, what does it mean for x with respect to A?

b) Deduce from the Borsuk-Ulam theorem that for any cover $\mathbb{S}^2=A_1\cup A_2\cup A_3$ by three non-empty closed subsets, one of the subsets A_i contains an antipodal pair $(x,-x)$.

c) For every covering $\mathbb{S}^2=U_1\cup U_2\cup U_3$ by three non-empty open subsets, one of the subsets U_i contains an antipodal pair $(x,-x)$.

   Hint: Either deduce it from b) using that $\mathbb{S}^2$ is compact, so locally compact (or sequentially compact), or use Borsak-Ulam again.
